Coherent States and Coordinate-Free Quantization *, **

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The usual quantization procedures interpret canonical transformations in an active way linking them with unitary transformations, while the quantization procedure offered by coherent states completely separates classical canonical transformations and unitary operator transformations. By exploiting this property, along with a physically motivated shadow metric, it is seen how to realize the quantization process in as coordinate-free a form as holds in classical mechanics.

Introduction

The inner beauty of classical mechanics as a theory cannot be denied, and the author is happy to join those who have professed an "affair of the heart" with this formalism [1]. One of the most beautiful aspects of classical mechanics is surely the Hamiltonian formulation and its invariance under canonical coordinate transformations. Indeed, this very invariance is a reflection of the existence of a deeper geometrical structure – symplectic geometry – that underlies the coordinate invariance [2] (much as a Riemannian geometry ensures coordinate invariance in that realm). How natural it would be, then, for quantum mechanics also to exhibit a corresponding invariance (or at least covariance) under canonical transformations of coordinates. To be sure, quantum mechanics does exhibit an invariance of its own, namely an invariance under unitary transformations of the Hilbert space basis. Some authors assert that invariance under unitary basis transformations is a direct quantum reflection of the invariance under canonical coordinate transformations in the classical theory [3]. This view arises naturally when one adopts a *direct* quantization prescription in which the classical canonical variables (p, q) are "promoted" to canonical (self-adjoint) operators (P, Q) that satisfy the usual commutator $[Q, P] = i(\hbar = 1)$. As a consequence the simple linear combinations $(p - q)/\sqrt{2}$ and $(q + p)/\sqrt{2}$ of classical variables are associated with the corresponding combinations $(P - Q)/\sqrt{2}$ and $(Q + P)/\sqrt{2}$ of quantum variables. However, the pair of variables $\bar{p} = (p - q)/\sqrt{2}$ and $\bar{q} = (q + p)/\sqrt{2}$ can also be considered to arise from a classical canonical transformation of the original pair, and it is natural to associate to this canonical transformation a corresponding unitary transformation that maps P and Q into $\bar{P} = (P - Q)/\sqrt{2}$ and $\bar{Q} = (Q + P)/\sqrt{2}$ which also satisfies the usual commutator $[\bar{Q}, \bar{P}] = i$. This direct association of canonical and unitary transformations is satisfactory for linear transformations of the kind illustrated but proves unsatisfactory for many nonlinear transformations. Consider the classical canonical transformation given by $\tilde{p} = (p^2 + q^2)/2$ and $\tilde{q} = \tan^{-1}(q/p)$ for which $\tilde{p} \geq 0$. At the very least a corresponding quantum pair $\tilde{P}$ and $\tilde{Q}$ ought to satisfy $\tilde{P} \geq 0$ as well as $[\tilde{Q}, \tilde{P}] = i$. However, it is known that there is no canonical pair of operators which are both self adjoint when the spectrum of one of them is semibounded. This may seem like a minor and irrelevant technicality until it is well understood that an operator being self adjoint or not corresponds to that operator being an observable or not; indeed, there is real physics in this "minor technicality"! Note well that for unbounded operators a Hermitian operator is not necessarily self adjoint, and it is the latter and more stringent property that is necessary for an operator to correspond to an observable.

The difficulty with the foregoing treatment, as we have presented it, is the adherence to an active interpretation of classical canonical transformations as opposed to a passive interpretation. In an active transformation the point in question in phase space is moved against a fixed system of coordinates (much as exhibited by the temporal evolution brought about by

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a nonvanishing Hamiltonian). In a passive transformation, on the other hand, the point in question in phase space remains unchanged, rather the canonical coordinate chart itself is changed (much as coordinate changes in a Riemannian geometry are generally viewed). As illustrated above, the usual quantization procedure (promoting q to the operator Q, etc.) inevitably leads to an active interpretation of canonical transformations. To reinterpret canonical transformations as passive transformations we will need to exhibit an alternative quantization procedure that completely decouples the canonical transformations of the classical theory from the unitary transformations of the quantum theory. Such an alternative quantization procedure is implicit in a view toward quantization that is offered by *coherent states* [4]. The approach developed in the next two sections not only achieves the goal of a quantization scheme that possesses the invariances of classical mechanics, but it does so in a way that simultaneously provides a rigorous mathematical formulation of the quantum mechanical path integral as well [5].

Coherent States

In the simple cases of interest to us here the coherent states are, first of all, unit vectors in Hilbert space that are in a continuous one-to-one correspondence with points of the classical phase space. For a single degree of freedom, phase space is two dimensional and thus the set of coherent states forms a smooth two-dimensional manifold in Hilbert space. In order to identify the phase-space points we lay down a set of canonical coordinates, and for each point we use the same coordinate values to identify the image vector in the set of coherent states. In symbols, if (p, q) denotes the two real coordinates of a phase-space point, then we denote by $|p, q\rangle$ the coherent state associated with that point. Moreover this association must be continuous, namely if $(p, q) \rightarrow (p', q')$ as real numbers, then $|p, q\rangle \rightarrow |p', q'\rangle$ in the sense of strong convergence; stated otherwise, the coherent-state overlap function $\langle p_1, q_1 | p_2, q_2\rangle$ is required to be jointly continuous in both label sets. Thus a continuous path $(p(t), q(t))$, $t' \leq t \leq t'' \equiv T + t'$, in phase space also corresponds to a continuous path $|p(t), q(t)\rangle$ in Hilbert space. Finally, we require that a positive measure $\mathrm{d}\mu(p, q)$ on phase space exists such that

$$I = \int |p, q\rangle\langle p, q| \,\mathrm{d}\mu(p, q)\,,$$

namely, that the identity operator I admits a continuous resolution in terms of one-dimensional projection operators $|p, q\rangle\langle p, q|$. If (p, q) corresponds to canonical coordinates, then there is one natural measure on phase space, the Liouville measure, and we seek sets of coherent states such that $\mathrm{d}\mu = C\,\mathrm{d}p\,\mathrm{d}q$ for some positive constant C.

There are many solutions to this limited set of conditions even for coherent states based on canonical operators. Let P, Q be self-adjoint, irreducible Heisenberg operators, where $[Q, P] = i$ holds on a common domain on which they both are essentially self-adjoint. Next choose an arbitrary, normalized fiducial vector $|0\rangle$, and define

$$|p, q\rangle = e^{-iqP} e^{ipQ} |0\rangle$$

for all $(p, q) \in \mathbf{R}^2$. These then are the canonical coherent states as we shall use them; they differ by a phase factor from another commonly used definition, namely

$$e^{i(pQ - qP)} |0\rangle = e^{ipq/2} e^{-iqP} e^{ipQ} |0\rangle\,.$$

Of course, any such phase factor disappears when the projection operator $|p, q\rangle\langle p, q|$ is considered. The operator

$$\int |p, q\rangle\langle p, q| \,\mathrm{d}p\,\mathrm{d}q$$

has the property that it commutes with the unitary operator $e^{-isP} e^{irQ}$ for all r, s, and by Schur's Lemma the integral in question must be proportional to the identity operator. An explicit calculation leads to the well-known form of the resolution of unity given by

$$I = \int |p, q\rangle\langle p, q| \,\mathrm{d}p\,\mathrm{d}q/2\pi\,,$$

which holds true for any normalized fiducial vector $|0\rangle$ [6].

To exhibit additional properties of the canonical coherent states it is expedient to impose next a very weak restriction on the fiducial vector, namely that

$$\langle 0|P|0\rangle = 0\,, \quad \langle 0|Q|0\rangle = 0\,.$$

It then follows that

$$\langle p, q|P|p, q\rangle = p\,, \quad \langle p, q|Q|p, q\rangle = q\,,$$

showing that the coherent state labels, p and q, just represent the *mean* values of the Heisenberg operators P and Q, and are not eigenvalues of any operator. More generally, the mean value of an operator $\mathscr{H} = \mathscr{H}(P, Q)$ is given by

$$\langle p, q|\mathscr{H}(P, Q)|p, q\rangle = \langle 0|\mathscr{H}(P + p, Q + q)|0\rangle \equiv H(p, q)\,.$$

For $\mathscr{H}$ an arbitrary polynomial it is known that $\mathscr{H}$ is uniquely determined by its diagonal matrix elements $H(p, q)$ for any choice of the fiducial vector [7]. Of course, the indicated evaluation depends heavily on the chosen parameterization of the coherent states; under a coordinate transformation these mean values will change.

Let us consider a transformation from the canonical variables (p, q) to another set of canonical variables $(\bar{p}, \bar{q})$. For the single degree of freedom under consideration a canonical transformation requires only that the Jacobian of the transformation is unity. Since the coherent states are in one-to-one correspondence with the points of phase space, then under such a coordinate change each one of the coherent states is *unchanged*, only its *label* is changed, in general. Specifically,

$$|p, q\rangle = |\bar{p}, \bar{q}\rangle \equiv \exp[-i q(\bar{p}, \bar{q}) P] \exp[i p(\bar{p}, \bar{q}) Q] |0\rangle ;$$

observe that the Dirac notation effectively forces us into a notational abuse here inasmuch as the functional dependence of $|\bar{p}, \bar{q}\rangle$ on $\bar{p}$ and $\bar{q}$ is in general quite different from the dependence of $|p, q\rangle$ on p and q. If instead we had denoted $|p, q\rangle$ by $\Phi[p, q]$ (as was the author's custom some years ago [6–8]), then after the canonical coordinate transformation we would have $\bar{\Phi}[\bar{p}, \bar{q}] = \Phi[p, q]$, emphasizing that although the coordinates and the functional dependence have changed the coherent state vectors themselves have remained invariant. With this notational point understood, it follows, after a canonical coordinate change, that

$$\begin{aligned} I &= \int |\bar{p}, \bar{q}\rangle \langle \bar{p}, \bar{q}| \, \mathrm{d}\bar{p} \, \mathrm{d}\bar{q}/2\pi , \\ \langle \bar{p}, \bar{q}| P |\bar{p}, \bar{q}\rangle &= p(\bar{p}, \bar{q}) , \\ \langle \bar{p}, \bar{q}| Q |\bar{p}, \bar{q}\rangle &= q(\bar{p}, \bar{q}) , \\ \langle \bar{p}, \bar{q}| \mathscr{H}(P, Q) |\bar{p}, \bar{q}\rangle &= \bar{H}(\bar{p}, \bar{q}) \equiv H(p(\bar{p}, \bar{q}), q(\bar{p}, \bar{q})) . \end{aligned}$$

The important point to observe here is that the classical coordinate transformations are completely *passive*; no change of the coherent-state vectors, and certainly no change of the quantum operators, occurs.

It is now perfectly acceptable to use the (polar-like) canonical coordinates $\tilde{p}$ and $\tilde{q}$ introduced earlier, so that, for example, the expectation

$$\langle p, q | \tfrac{1}{2}(P^2 + Q^2 - c) | p, q\rangle = \tfrac{1}{2}(p^2 + q^2) ,$$

where $c = \langle 0 | (P^2 + Q^2) | 0\rangle$, reads, after the transformation, as

$$\langle \tilde{p}, \tilde{q} | \tfrac{1}{2}(P^2 + Q^2 - c) | \tilde{p}, \tilde{q}\rangle = \tilde{p} ,$$

while the operator whose mean is so evaluated remains absolutely unchanged by this coordinate transformation. Of course, there is a completely separate covariance of the identity operator and invariance of the expectation values under an arbitrary but simultaneous unitary transformation of the coherent state vectors and the operators of the form

$$\begin{aligned} |p, q\rangle &\to U |p, q\rangle , \\ P &\to U P U^{\dagger} , \\ Q &\to U Q U^{\dagger} , \\ \mathscr{H}(P, Q) &\to U \mathscr{H}(P, Q) U^{\dagger} . \end{aligned}$$

Further evidence of the passive nature of canonical coordinate transformations when coherent states are involved may be found if the fiducial vector satisfies the equation $(Q + iP)|0\rangle = 0$; observe that in making this choice we have conveniently chosen units so that P and Q appear to have the same dimensions. With this choice of fiducial vector then it is known that

$$H(p, q) = \langle p, q | \mathscr{H}(P, Q) | p, q\rangle$$

uniquely determines a general operator $\mathscr{H}$ even when it is no longer restricted to be a polynomial [8]. In addition, a wholly different operator representation opens up in the form [9, 10]

$$\mathscr{H} = \int h(p, q) |p, q\rangle \langle p, q| \, \mathrm{d}p \, \mathrm{d}q/2\pi .$$

This prescription (weakly) defines an operator for any function h that is polynomially bounded, and such operators are dense in the set of all operators in almost any desired sense. This diagonal coherent-state representation of operators was discovered in 1963 by George Sudarshan in the context of quantum optics [9], and it is the basis of a number of the quasi-classical formulations of quantum optics.

Under a canonical coordinate transformation from (p, q) to $(\bar{p}, \bar{q})$ it follows that

$$\mathscr{H} = \int \bar{h}(\bar{p}, \bar{q}) \bar{p}, \bar{q}\rangle \langle \bar{p}, \bar{q}| \, \mathrm{d}\bar{p} \, \mathrm{d}\bar{q}/2\pi ,$$

namely, the diagonal weight function has changed its dependence on the coordinates according to the relation

$$\bar{h}(\bar{p}, \bar{q}) \equiv h(p(\bar{p}, \bar{q}), q(\bar{p}, \bar{q})) ,$$

but neither the coherent-state projection operators nor the represented operator $\mathscr{H}$ have changed at all. On the other hand, the diagonal coherent-state representation is covariant under a simultaneous unitary transformation of the coherent states and the represented operator of the form $|p, q\rangle \to U |p, q\rangle$ and $\mathscr{H} \to U \mathscr{H} U^{\dagger}$.

We next observe that, if $\mathscr{H}$ is Hermitian then both $h(p, q)$ and $H(p, q)$ are *real*, and this property is preserved under (real) canonical coordinate and unitary operator transformations. In general, $h(p, q)$ and $H(p, q)$ differ from one another but the difference terms are $O(\hbar)$ and thus vanish in the limit that $\hbar \to 0$. Typically the $O(\hbar)$ terms are extremely small and do not significantly affect either the quantitative or qualitative properties of either $h(p, q)$ or $H(p, q)$. We conclude from this that we can safely interpret the physical meaning of either $h(p, q)$ or $H(p, q)$ as the physical meaning of the expression that results in the limit $\hbar \to 0$. When $\mathscr{H}(P, Q)$ is the Hamiltonian operator, the limiting expression is the classical Hamiltonian, and we thus conclude that either $h(p, q)$ or $H(p, q)$, depending on the circumstances, can serve in the role of classical Hamiltonian. These expressions may indeed depend on $\hbar$, but in this context $\hbar$ is just an extra parameter in the classical theory having no other significance. Finally, we note the general relation between $h(p, q)$ and $H(p, q)$ in the special coordinates and for the harmonic oscillator ground state fiducial vector, namely,

$$h(p, q) = e^{-\hbar(\partial^2/\partial p^2 + \partial^2/\partial q^2)/2} H(p, q),$$

where in this case the dependence on $\hbar$ of the distinguishing operation is explicitly given. Stated alternatively, in the present case $H(p, q)$ is the normally ordered symbol of $\mathscr{H}$, while $h(p, q)$ is the anti-normally ordered symbol of $\mathscr{H}$ [10].

Quantum and Classical Action Principles

We close this section by offering once again an old application of the forgoing ideas in which $H(p, q)$ is the natural candidate to choose as the classical Hamiltonian [4, 6]. Recall the action functional for quantum mechanics given by ($\hbar = 1$ again)

$$I_Q = \int \left[i \left\langle \psi(t) \left| \frac{\partial}{\partial t} \right| \psi(t) \right\rangle - \langle \psi(t) | \mathscr{H} | \psi(t) \rangle \right] dt .$$

Extremizing this expression over variations of $\langle \psi(t)|$ that vanish at the end points leads to the Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \mathscr{H} |\psi(t)\rangle .$$

On the other hand, what is the consequence of making a *restricted* set of variations? In particular, let $|\psi(t)\rangle = |p(t), q(t)\rangle$ denote the limited form vectors can assume, in which case

$$I_Q|_{|\psi\rangle = |p, q\rangle} = \int \left[i \left\langle p(t), q(t) \left| \frac{\partial}{\partial t} \right| p(t), q(t) \right\rangle - \langle p(t), q(t) | \mathscr{H} | p(t), q(t) \rangle \right] dt .$$

It is a simple exercise to show that

$$i \left\langle p(t), q(t) \left| \frac{\partial}{\partial t} \right| p(t), q(t) \right\rangle = p(t) \frac{d}{dt} q(t)$$

holds for any choice of $|0\rangle$ that satisfies the weak conditions $\langle 0|P|0\rangle = \langle 0|Q|0\rangle = 0$. Consequently,

$$I_Q|_{|\psi\rangle = |p, q\rangle} = \int [p(t)\dot{q}(t) - H(p(t), q(t))] \, dt ,$$

which is nothing but the *classical* action, I_C, the extremal variation of which, holding the end points fixed, leads to the classical Hamiltonian equations

$$\dot{q}(t) = \partial H/\partial p(t) ,$$
$$\dot{p}(t) = -\partial H/\partial q(t) .$$

Stated otherwise, there is only *one* action principle (I_Q) in physics, that for quantum mechanics; by suitably restricting its domain, the quantum action ($I_Q|_{|\psi\rangle = |p, q\rangle}$) equals the classical action (I_C). Thus, a partial degree of quantization is already attained merely by *reinterpreting* any classical action as the restricted version of an appropriate quantum action [7]!

The result that restricting the quantum action to coherent states results in the classical action applies to a wide variety of physical systems and their associated coherent states. For two recent articles utilizing this general viewpoint see [11] and [12].

Coordinate-Free Quantization

Classical Mechanics

In this section we adopt

$$I = \int [p \, dq + dG(p, q) - h(p, q) \, dt]$$

as the classical action from which the Hamiltonian equations

$$\dot{q}(t) = \partial h/\partial p(t) ,$$
$$\dot{p}(t) = -\partial h/\partial q(t)$$

follow from extremal variations that hold the endpoints fixed. Since the endpoints are fixed, the same equations of motion arise for any choice of G. Thus there is an *equivalence class* of actions all of which lead to the same equations of motion. We have chosen the function h to represent the classical Hamiltonian since we will soon identify h as the weight function in the diagonal coherent-state representation of the quantum Hamiltonian.

A canonical coordinate transformation implies the existence of a function $\bar{F}$ such that

$$p\,\mathrm{d}q = \bar{p}\,\mathrm{d}\bar{q} + \mathrm{d}\bar{F}(p, \bar{q})\,.$$

Given that the classical Hamiltonian transforms as a scalar,

$$\bar{h}(\bar{p}, \bar{q}) = h(p, q)\,,$$

it is clear that there exists a function $\bar{G}$ which incorporates the effects of G and $\bar{F}$, so that

$$I = \int [\bar{p}\,\mathrm{d}\bar{q} + \mathrm{d}\bar{G}(\bar{p}, \bar{q}) - \bar{h}(\bar{p}, \bar{q})]\,\mathrm{d}t\,.$$

Thus the equivalence class of classical actions is *invariant* under canonical coordinate transformations. Clearly extremal variations holding the endpoints fixed lead to the Hamiltonian equations

$$\dot{\bar{q}}(t) = \partial\bar{h}/\partial\bar{p}(t)\,,$$
$$\dot{\bar{p}}(t) = -\,\partial\bar{h}/\partial\bar{q}(t)\,,$$

which are therefore identical in form in any canonical coordinate system.

This invariance reflects the underlying symplectic geometry of classical mechanics. Symplectic geometry is characterized by a (phase-space) manifold M and a symplectic form ω, a closed nondegenerate two form [2]. Since ω is closed, then it is the exterior derivative (d) of a one form θ, $\omega = \mathrm{d}\theta$, at least locally. This one form θ is not unique, and various one forms $\theta + \mathrm{d}G$ for arbitrary G are equally acceptable since $\mathrm{d}(\theta + \mathrm{d}G) = \omega$ just as well. With h chosen as a scalar on M it follows that

$$I = \int [\theta + \mathrm{d}G - h\,\mathrm{d}t]$$

provides a coordinate-free characterization of each action in the equivalence class. As far as possible, it is just this kind of coordinate-free description we seek to find in the quantum theory.

Shadow Metric

As we have seen earlier, in one set of canonical coordinates a harmonic oscillator reads $(p^2 + q^2)/2$, in another set it reads $\tilde{p}$. How is one to know when a given coordinate expression refers to a harmonic oscillator or to some other physical system without recourse to a "coordinate-free oscillator"? We claim that one needs to have a *shadow metric* $\mathrm{d}\sigma^2$ on a flat two-dimensional replica of phase space with which one is able to keep track of the various canonical coordinate systems including the special Cartesian ones. For example, if the coordinates of the shadow metric are chosen so that $\mathrm{d}\sigma^2 = \mathrm{d}p^2 + \mathrm{d}q^2$, then one is assured that an expression like $\frac{1}{2}(p^2 + q^2) + \lambda q^4$ which *looks* like a quartic anharmonic oscillator actually *refers* to a genuine, physical quartic anharmonic oscillator, or $\frac{1}{2}(p^2 + q^2)$ which *looks* like an harmonic oscillator *is* a harmonic oscillator. Moreover, after a certain canonical coordinate change $(p, q) \to (\tilde{p}, \tilde{q})$, as described above, then $\frac{1}{2}(p^2 + q^2) = \tilde{p}$, and to recognize that $\tilde{p}$ actually refers to a harmonic oscillator one need only observe that the shadow metric now reads $\mathrm{d}\sigma^2 = \mathrm{d}\tilde{p}^2/(2\tilde{p}) + (2\tilde{p})\,\mathrm{d}\tilde{q}^2$. In brief, the *physics* underlying the form invariance of Hamilton's equations is coded into a shadow metric on a flat phase space – "shadow" because it does not appear in the formulation of Hamilton's equations. Thus the shadow metric implicitly keeps track of what *physical* system the *mathematical* expression for the Hamiltonian refers to.

Continuous-Time Regularization of Path Integrals

It is probably true that most readers will have encountered the formal path integral expression

$$\mathcal{M} \int e^{i\int [p\dot{q} + \dot{G}(p,q) - h(p,q)]\,\mathrm{d}t}\,\mathcal{D}p\,\mathcal{D}q\,,$$

perhaps without the $\dot{G}$ term, and again most readers have been sufficiently well conditioned as to know what kind of quantum-mechanical matrix element it presumably represents. It is clear that this formal path integral is incomplete by itself since this expression is formally covariant (invariant if G is changed as well) under a canonical coordinate transformation. If, instead, the formal path integral were complete, a complicated classical Hamiltonian could be rendered simple merely by a coordinate change, and this would have the consequence that the spectrum of the quantum Hamiltonian would also be changed. To prevent such misapplications of the formal path integral the shadow metric needs to be invoked. In a lattice regularization of the formal path integral, canonical coordinates must be identified in which a natural and naive lattice action is correct; these coordinates are just Cartesian coordinates of the shadow metric. Indeed most derivations of the phase-space path integral proceed from the operator formulation and derive an acceptable lattice-space regularized form of the path integral that most people have already seen and probably have been preconditioned to substitute, at least mentally, for the formal path integral itself. We ask the reader to please suspend any preconceived interpretation and instead just accept the formal path integral as it stands, as we now introduce a less-than-standard regularization that also invokes the shadow metric in order to make it well defined.

Let us initially interpret the formal path integral as

$$\lim_{v\to\infty} \mathscr{M} \int e^{i\int[p\dot q + \dot G(p,q) - h(p,q)]\,dt}\, e^{-\frac{1}{2v}\int(\dot p^2+\dot q^2)\,dt}\, \mathscr{D}p\,\mathscr{D}q\,.$$

Observe first that the v-dependent factor in the integrand formally goes to unity as $v\to\infty$, leading to something like the original expression. Observe also that the v-dependent factor involves a flat phase-space metric – *the shadow metric itself* – here expressed in Cartesian coordinates, $dp^2 + dq^2$. The inserted factor, together with a formal factor from $\mathscr{M}$ and the formal flat measure may be replaced by Wiener measure on phase space. Thus we further refine our interpretation of the path integral by replacing the previous expression by

$$\lim_{v\to\infty} 2\pi e^{vT/2} \int e^{i\int[p\,dq + dG(p,q) - h(p,q)\,dt]}\, d\mu_W^v(p,q)\,,$$

where we have anticipated the proper prefactors and have denoted by μ_W^v a Wiener measure describing Brownian motion on a flat two-dimensional space with diffusion constant v.

In the latter form the paths $p(t)$ and $q(t)$ are explicitly Brownian in character, and as such the integral $\int p(t)\,dq(t)$ cannot be defined as a conventional Stieltjes integral (because Brownian motion paths have unbounded variation). However, the integral can be well defined as a *stochastic integral* [13], which we adopt in the Stratonovich form

$$\int p(t)\,dq(t) = \lim_{\varepsilon\to 0} \sum \tfrac{1}{2}(p_{k+1}+p_k)(q_{k+1}-q_k)\,,$$

where $q_k = q(k\varepsilon + t')$, $k = 0, 1, 2, \ldots, K = T/\varepsilon$, etc. Wiener measure needs to be pinned somewhere, say at the initial time t' at $p(t') = p'$ and $q(t') = q'$. However, since the complex conjugate expression must correspond to the time-reversed matrix element, it follows that if the Wiener measure is pinned at the initial time t' it must also be pinned at the final time $t'' = T + t'$, $T > 0$, say at $p(t'') = p''$ and $q(t'') = q''$. Thus the Wiener-measure regularized path integral is well defined for all $v < \infty$, depends on four variables and two times (or one time difference), and is of the form $K(p'', q'', t''; p', q', t')$.

As $v\to\infty$ one must determine whether or not the limit exists, and if so does the limit have anything to do with solutions of the Schrödinger equation. Affirmative answers to both questions have been provided in [14], and here we quote only the result

$$K(p'', q'', t''; p', q', t') = \langle p'', q''|\, e^{-i\mathscr{H}T}\, |p', q'\rangle$$
$$= \lim_{v\to\infty} 2\pi e^{vT/2} \int e^{i\int[p\,dq + dG(p,q) - h(p,q)\,dt]}\, d\mu_W^v(p,q)\,,$$

where

$$|p,q\rangle \equiv e^{-iG(p,q)} e^{-iqP} e^{ipQ}|0\rangle\,, \quad (Q + iP)|0\rangle = 0\,,$$
$$\mathscr{H} = \int h(p,q)\,|p,q\rangle\langle p,q|\,dp\,dq/2\pi$$

just as before (save for the phase G). This result holds for all classical Hamiltonians h that have global classical solutions.

Although the path integral involves Brownian motion paths, the Stratonovich prescription ensures that the ordinary rules of calculus are still satisfied [13]. Thus a canonical coordinate change is still given by $p(t)\,dq(t) = \bar p(t)\,d\bar q(t) + d\bar F(q(t), \bar q(t))$, and the same function $\bar G$ exists so that, after such a change, the Wiener-measure regularized expression reads

$$\bar K(\bar p'', \bar q'', t''; \bar p', \bar q', t') = \langle \bar p'', \bar q''|\, e^{-i\mathscr{H}T}\, |\bar p', \bar q'\rangle$$
$$= \lim_{v\to\infty} 2\pi e^{vT/2} \int e^{i\int[\bar p\,d\bar q + d\bar G(\bar p,\bar q) - \bar h(\bar p,\bar q)\,dt]}\, d\bar\mu_W^v(\bar p,\bar q)\,.$$

Observe that, just as stressed in the previous section, the coordinate transformation is passive so that, in particular, the Hamiltonian operator $\mathscr{H}$ has remained unchanged.

Finally, the propagator can be given a coordinate-free formulation in the form

$$K(t''; t') = \lim_{v\to\infty} 2\pi e^{vT/2} \int e^{i\int(\theta + dG - h\,dt)}\, d\mu_W^v\,.$$

Here $K(t''; t'): M \times M \to \mathbf{C}$ and μ_W^v denotes an intrinsically defined Brownian motion that also depends on $M \times M$ since it is pinned initially and finally. Observe if $h = 0$ that the propagator reduces to the reproducing kernel $\mathscr{K}: M \times M \to \mathbf{C}$ given by

$$\mathscr{K} = \lim_{v\to\infty} 2\pi e^{vT/2} \int e^{i\int(\theta + dG)}\, d\mu_W^v\,;$$

in coordinates, the reproducing kernel just corresponds to the coherent-state overlap function. In coordinate-free form the propagator $K(t; t')$ is a solution of the Schrödinger equation [5, 14]

$$i\,\frac{\partial\phi(t)}{\partial t} = \mathscr{H}\,\phi(t)\,,$$

where

$$\mathscr{H} = \mathscr{K}\,h\,\mathscr{K}\,,$$

subject to the boundary condition that

$$\lim_{t\to t'} K(t;t') = \mathscr{K}\,.$$

We close by emphasizing that the continuous-time, Wiener-measure regularized path integral not only offers a coordinate-free formulation of the quantization process itself, but at the same time provides a rigorous mathematical formulation of path-integral quantization involving genuine (countably additive) measures on the space of continuous functions.

Finally, we note that additional applications of these ideas have been presented elsewhere to the quantization of spin systems [14], to the quantization of pseudo-spin systems [15], and even to a speculative, but nevertheless natural, quantization proposal for the gravitational field [16].

Dedication

It is a pleasure to dedicate this article to the 60th birthday of George Sudarshan. His gentle manner and remarkable creativity have both been persuasive forces among his admirers and his followers.

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